Appendix: A New Fact to Discipline Models of Beliefs

Julio Ortiz

March 2020

Contents

Appendix A Derivations

A.1 Deriving Errors and Revisions for SI Model

In state space form, we have

$$\begin{bmatrix} x_t \\ F_t \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ \lambda\rho & (1-\lambda)\rho \end{bmatrix} \begin{bmatrix} x_{t-1} \\ F_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ \lambda \end{bmatrix} w_t$$

$$y_t^i = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ F_t \end{bmatrix} + v_t^i$$

Or more compactly,

$$\mathbf{s_t} = \mathbf{A}\mathbf{s_{t-1}} + \mathbf{B}w_t$$

$$y_t^i = \mathbf{C}\mathbf{s_t} + v_t^i$$

Invoking the Kalman filter

$$\mathbf{s^i_{t|t}} = (\mathbf{I} - \kappa\mathbf{C})\mathbf{A}\mathbf{s^i_{t-1|t-1}} + \kappa\mathbf{C}\mathbf{A}\mathbf{s_{t-1}} + \kappa\mathbf{C}\mathbf{B}w_t + \kappa v_t^i$$

$$\mathbf{s^i_{t|t-1}} = \mathbf{A}\mathbf{s^i_{t-1|t-1}}$$

which implies that

$$x^i_{t|t} = (1-\kappa_1)\rho x^i_{t-1|t-1} + \kappa_1 y_t^i$$

$$F^i_{t|t} = (\lambda - \kappa_2)\rho x^i_{t-1|t-1} + (1-\lambda)\rho F^i_{t-1|t-1} + \kappa_2 y_t^i$$

and

$$x^i_{t|t-1} = \rho x^i_{t-1|t-1}$$

$$F^i_{t|t-1} = \lambda\rho x^i_{t-1|t-1} + (1-\lambda)\rho F^i_{t-1|t-1}$$

where $\lambda = \frac{\kappa_1 + R\kappa_2}{1+R}$ and κ_1, κ_2 are the Kalman gains belonging to the two-dimensional column vector κ. Letting $\xi = \begin{bmatrix} \frac{1}{1+R} & \frac{R}{1+R} \end{bmatrix}$, we have

$$\begin{aligned}
\widetilde{x}^i_{t|t} &= \xi \mathbf{s^i_{t|t}} \\
&= \frac{1}{1+R} x^i_{t|t} + \frac{R}{1+R} F^i_{t|t} \\
&= \frac{1}{1+R}\Big[(1-\kappa_1)\rho x^i_{t-1|t-1} + \kappa_1 y_{it}\Big] + \frac{R}{1+R}\Big[(\lambda - \kappa_2)\rho x^i_{t-1|t-1} + (1-\lambda)\rho F^i_{t-1|t-1} + \kappa_2 y_{it}\Big] \\
&= \frac{1-\kappa_1 + R(\lambda-\kappa_2)}{1+R} x^i_{t|t-1} + \lambda y_{it} + \frac{(1-\lambda)R\rho}{1+R} F^i_{t-1|t-1} \\
&= \frac{1-\lambda}{1+R} x^i_{t|t-1} + \frac{(1-\lambda)R\rho}{1+R} F^i_{t-1|t-1} + \lambda y_{it}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{x}^i_{t|t-1} &= \xi \mathbf{s^i_{t|t-1}} \\
&= \frac{1}{1+R}\rho x^i_{t-1|t-1} + \frac{R}{1+R}\Big[\lambda\rho x^i_{t-1|t-1} + (1-\lambda)\rho F^i_{t-1|t-1}\Big] \\
&= \frac{1+R\lambda}{1+R} x^i_{t|t-1} + \frac{(1-\lambda)\rho R}{1+R} F^i_{t-1|t-1}
\end{aligned}$$

Hence,

$$\begin{aligned}
\widetilde{x}^i_{t|t} &= \frac{1-\lambda}{1+R} x^i_{t|t-1} + \frac{(1-\lambda)R\rho}{1+R} F^i_{t-1|t-1} + \lambda y^i_t \\
\widetilde{x}^i_{t|t-1} &= \frac{1+R\lambda}{1+R} x^i_{t|t-1} + \frac{(1-\lambda)\rho R}{1+R} F^i_{t-1|t-1}
\end{aligned}$$

Furthermore,

$$F_t = \widetilde{x}_{t|t} = (1-\lambda)\rho F_{t-1} + \lambda x_t$$

$$\widetilde{x}_{t|t-1} = \lambda\rho x_{t-1|t-1} + (1-\lambda)\rho F_{t-1}$$

From here, it follows that

$$\begin{aligned}
x_t - \widetilde{x}^i_{t|t} &= x_t - \frac{1-\kappa_1+(\lambda-\kappa_2)R}{1+R}x^i_{t|t-1} - \lambda y^i_t - \frac{(1-\lambda)R\rho}{1+R}F^i_{t-1|t-1} \\
&= (1-\lambda)x_t - \frac{1-\kappa_1+(\lambda-\kappa_2)R}{1+R}x^i_{t|t-1} - \lambda v^i_t - \frac{(1-\lambda)R\rho}{1+R}F^i_{t-1|t-1} \\
&= (1-\lambda)x_t - \frac{1-\lambda}{1+R}x^i_{t|t-1} - \frac{(1-\lambda)R\rho}{1+R}F^i_{t-1|t-1} - \lambda v^i_t \\
&= (1-\lambda)\left[x_t - \frac{1}{1+R}x^i_{t|t-1} - \frac{R}{1+R}\rho F^i_{t-1|t-1}\right] - \lambda v^i_t
\end{aligned}$$

and the forecast revision is

$$\begin{aligned}
\widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1} &= \frac{(1-\kappa_1)+(\lambda-\kappa_2)R-(1+R\lambda)}{1+R}x^i_{t|t-1} + \lambda x_t + \lambda v^i_t \\
&= -\frac{\kappa_1-\kappa_2 R}{1+R}x^i_{t|t-1} + \lambda x_t + \lambda v^i_t \\
&= \lambda(x_t - x^i_{t|t-1} + v^i_t)
\end{aligned}$$

A.2 Proof of Proposition 2

$$\begin{aligned}
\beta_1^{SI} &= \frac{\text{Cov}(x_t - \widetilde{x}^i_{t|t}, \widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1})}{\text{Var}(\widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1})} \\
&= \frac{(1-\lambda)\lambda \text{Cov}(x_t, x_t - x^i_{t|t-1}) - \lambda^2 \sigma_v^2}{\lambda^2 \left[\text{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2 \right]} \\
&= \frac{1-\lambda}{\lambda} \frac{\text{Var}(x_t) - \text{Cov}(x_t, x^i_{t|t-1})}{\text{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2} - \frac{\sigma_v^2}{\text{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2} \\
&= \frac{1-\lambda}{\lambda} \frac{\frac{\sigma_w^2}{1-\rho^2} - \frac{\rho^2 \kappa_1 \cdot \frac{\sigma_w^2}{1-\rho^2}}{1-(1-\kappa_1)\rho^2}}{\text{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2} - (1-\kappa_1) \\
&= \frac{1-\lambda}{\lambda} \frac{\frac{\sigma_w^2}{1-\rho^2} \frac{1-\rho^2}{1-(1-G)\rho^2}}{\text{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2} - (1-\kappa_1) \\
&= \frac{1-\lambda}{\lambda} \frac{\text{Var}(x_t - x^i_{t|t-1})}{\text{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2} - (1-\kappa_1) \\
&= \frac{1-\lambda}{\lambda} \kappa_1 - (1-\kappa_1) \\
&= \frac{1-\kappa_1 + R(1-\kappa_2)}{\kappa_1 + R\kappa_2} \kappa_1 - (1-\kappa_1) \\
&= \frac{R(\kappa_1 - \kappa_2)}{\kappa_1 + R\kappa_2}
\end{aligned}$$

A.3 Equating β_1 Across Models

DE → SI

$\beta_1^{DE} = \frac{-\theta(1+\theta)}{(1+\theta)^2 + \rho^2\theta^2}$ and $\beta_1^{SI} = \frac{R(\kappa_1 - \kappa_2)}{\kappa_1 + R\kappa_2}$.

Given $\{\rho, \sigma_w, \sigma_v, \theta\}$ we solve for R by setting $\beta_1^{DE} = \beta_1^{SI}$

$$\begin{aligned}
\frac{R(\kappa_1 - \kappa_2)}{\kappa_1 + R\kappa_2} &= \beta_1^{DE} \\
R(\kappa_1 - \kappa_2) &= \beta_1^{DE}(\kappa_1 + R\kappa_2) \\
R(\kappa_1 - \kappa_2) &= \beta_1^{DE}\kappa_1 + \beta_1^{DE} R\kappa_2 \\
R(\kappa_1 - \kappa_2) - \beta_1^{DE} R\kappa_2 &= \beta_1^{DE}\kappa_1 \\
R &= \frac{\beta_1^{DE}\kappa_1}{\kappa_1 - (1 + \beta_1^{DE})\kappa_2}
\end{aligned}$$

SI→DE

Given $\{\rho, \sigma_w, \sigma_v, R\}$ we solve for θ by setting $\beta_1^{DE} = \beta_1^{SI}$

$$\begin{aligned}
\frac{-\theta(1+\theta)}{(1+\theta)^2 + \rho^2\theta^2} &= \beta_1^{SI} \\
-\theta(1+\theta) &= \beta_1^{SI}(1+\theta)^2 + \beta_1^{SI}\rho^2\theta^2 \\
-\theta - \theta^2 &= \beta_1^{SI}(1 + 2\theta + \theta^2) + \beta_1^{SI}\rho^2\theta^2 \\
0 &= \beta_1^{SI} + 2\beta_1^{SI}\theta + \theta + \beta_1^{SI}\theta^2 + \theta^2 + \beta_1^{SI}\rho^2\theta^2 \\
0 &= \beta_1^{SI} + (2\beta_1^{SI} + 1)\theta + [1 + (1+\rho^2)\beta_1^{SS}]\theta^2
\end{aligned}$$

A.4 Proof of Proposition 3

It suffices to show that $\mathrm{Cov}(\widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1}, \widetilde{x}^i_{t|t-1} - \widetilde{x}^i_{t|t-2}) = 0$. The current period revision is

$$\begin{aligned}
\widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1} &= \frac{1 - \lambda - 1 - R\lambda}{1+R} x^i_{t|t-1} + \lambda y_{it} \\
&= \lambda(y_{it} - x^i_{t|t-1})
\end{aligned}$$

and the previous period revision can be expressed as

$$\begin{aligned}
\widetilde{x}^i_{t|t-1} - \widetilde{x}^i_{t|t-2} &= \frac{1}{1+R}(x^i_{t|t-1} - x^i_{t|t-2}) + \frac{R}{1+R}(F^i_{t|t-1} - F^i_{t|t-2}) \\
&= \frac{\rho}{1+R}(x^i_{t-1|t-1} - x^i_{t-1|t-2}) + \frac{R}{1+R}\big[\lambda\rho x^i_{t-1|t-1} + (1-\lambda)\rho F^i_{t-1|t-1} - \lambda\rho x^i_{t-1|t-2} \\
&- (1-\lambda)\rho F^i_{t-1|t-2}\big] \\
&= \frac{\rho}{1+R}\big[\kappa_1(y_{it-1} - x^i_{t-1|t-2})\big] + \frac{R}{1+R}\big[\lambda\rho(x^i_{t-1|t-1} - x^i_{t-1|t-2}) + (1-\lambda)\rho(F^i_{t-1|t-1} - F^i_{t-1|t-2})\big] \\
&= \frac{\rho\kappa_1}{1+R}\big[x_{t-1} - x^i_{t-1|t-2} + v_{it-1}\big] + \frac{R}{1+R}\big[\lambda\rho\kappa_1(x_{t-1} - x^i_{t-1|t-2} + v_{it-1}) \\
&+ (1-\lambda)\rho\kappa_2(x_{t-1} - x^i_{t-1|t-2} + v_{it-1})\big] \\
&= \left[\frac{\rho\kappa_1 + R\lambda\rho\kappa_1 + R(1-\lambda)\rho\kappa_2}{1+R}\right](x_{t-1} - x^i_{t-1|t-2} + v_{it-1}) \\
&= \frac{\rho(\kappa_1 + R\kappa_2) + \rho R\lambda(\kappa_1 - \kappa_2)}{1+R}(x_{t-1} - x^i_{t-1|t-2} + v_{it-1}) \\
&= \rho\lambda\left(1 + \frac{R(\kappa_1 - \kappa_2)}{1+R}\right)(x_{t-1} - x^i_{t-1|t-2} + v_{it-1})
\end{aligned}$$

Then,

$$
\begin{aligned}
\mathrm{Cov}(\widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1}, \widetilde{x}^i_{t|t-1} - \widetilde{x}^i_{t|t-2}) &= \mathrm{Cov}\Bigg[\lambda(y_{it} - x^i_{t|t-1}), \rho\lambda\left(1 + \frac{R(\kappa_1 - \kappa_2)}{1+R}\right)(x_{t-1} - x^i_{t-1|t-2} + v_{it-1})\Bigg] \\
&= \mathrm{Cov}\Bigg[\lambda(1-\kappa_1)\rho(x_{t-1} - x^i_{t-1|t-2}) - \lambda\kappa_1\rho v_{it-1} + \lambda(v_{it} + w_t), \\
&\rho\lambda\left(1 + \frac{R(\kappa_1 - \kappa_2)}{1+R}\right)(x_{t-1} - x^i_{t-1|t-2} + v_{it-1})\Bigg] \\
&= (\lambda\rho)^2\left[1 + \frac{R(\kappa_1 - \kappa_2)}{1+R}\right]\left[(1-\kappa_1)\Psi - \kappa_1\sigma_v^2\right] \\
&= 0
\end{aligned}
$$

By definition, $\kappa_1 = \frac{\Psi}{\Psi + \sigma_v^2}$ where Ψ is the steady state forecast error variance (i.e. the variance that solves the Ricatti equation: $\Psi = (1-\kappa_1)\rho^2\Psi + \sigma_w^2$). From this it follows that the last term in hard brackets is zero.

Appendix B Overconfidence Model

Overconfidence Model

Daniel (1993) presents a theory of overconfidence in which individuals perceive their private signals to be more precise than they truly are. Specifically, forecasters perceive

$$y_t^i = x_t + v_t^i \qquad v_t^i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \breve{\sigma}_v^2)$$

where $\breve{\sigma}_v = \alpha\sigma_v$ such that $\alpha < 1$.

As a matter of notation, let the forecaster's current period forecast as $\breve{x}_{t|t}^i$ and his one step ahead forecast as $\breve{x}_{t|t-1}^i$. Forecasters invoke the Kalman filter for the model in order for formulate their expectations. As a result, expectations are determined according to the following predict-update procedure

$$\begin{aligned} \breve{x}_{t|t-1}^i &= \rho\breve{x}_{t-1|t-1}^i && \text{(Predict)} \\ \breve{x}_{t|t}^i &= \breve{x}_{t|t-1}^i + \breve{\kappa}(y_t^i - \breve{x}_{t|t-1}^i) && \text{(Update)} \end{aligned}$$

Proposition 1. *The OLS coefficient arising from an errors-on-revisions regression in the overconfidence model is*

$$\beta_1^{OC} = \frac{\mathbb{C}(x_t - \breve{x}_{t|t}^i, \breve{x}_{t|t}^i - \breve{x}_{t|t-1}^i)}{\mathbb{V}(\breve{x}_{t|t}^i - \breve{x}_{t|t-1}^i)} = \frac{(\alpha^2 - 1)\sigma_v^2}{\mathbb{V}(x_t - \breve{x}_{t|t-1}^i) + \sigma_v^2} < 0$$

Proof.

$$\begin{aligned}
\beta_1^{OC} &= \frac{\mathbb{C}(x_t - \breve{x}^i_{t|t}, \breve{x}^i_{t|t} - \breve{x}^i_{t|t-1})}{\mathbb{V}(\breve{x}^i_{t|t} - \breve{x}^i_{t|t-1})} \\
&= \frac{(1-\breve{\kappa})\breve{\kappa}\mathbb{V}(x_t - x^i_{t|t-1}) - \breve{\kappa}^2\sigma_v^2}{\breve{\kappa}^2[\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2]} \\
&= \frac{(1-\breve{\kappa})\mathbb{V}(x_t - x^i_{t|t-1}) - \breve{\kappa}\sigma_v^2}{\breve{\kappa}[\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2]} \\
&= \frac{\frac{\breve{\sigma}_v^2\mathbb{V}(x_t - \breve{x}^i_{t|t-1})}{\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \breve{\sigma}_v^2} - \frac{\mathbb{V}(x_t - \breve{x}^i_{t|t-1})\sigma_v^2}{\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \breve{\sigma}_v^2}}{\breve{\kappa}[\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2]} \\
&= \frac{(\alpha^2 - 1)\sigma_v^2\breve{\kappa}}{\breve{\kappa}[\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2]} \\
&= \frac{(\alpha^2 - 1)\sigma_v^2}{\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2}
\end{aligned}$$

□

The coefficient β_1^{OC} is always negative since $\alpha < 1$.[1] Hence, the underlying overconfidence on the part of forecasters generates overreactions. In essence, forecasters believe their signals to be more precise than they truly are thereby leading them to place more weight on new information.

B.1 Mapping Overconfidence to Diagnostic Expectations

A model of overconfidence can deliver the same β_1 coefficient as a model of diagnostic expectations. Since the variance of the error depends nonlinearly on α, this is done by finding the α parameter such that

$$\alpha^2 - \frac{\beta_1^{DE}}{\sigma_v^2}\breve{\Psi}^i_{t|t-1}(\alpha) = \beta^{DE} + 1$$

where $\breve{\Psi}^i_{t|t-1}(\alpha)$ is the forecast error variance in the overconfidence model. Note that this is itself a function of α.

[1] In the absence of overconfidence, $\alpha = 1$ and $\beta_1 = 0$.

Given $\{\rho, \sigma_w, \sigma_v, \theta\}$, we solve for α by setting $\beta_1^{DE} = \beta_1^{OC}$

$$\begin{aligned}
\frac{(\alpha^2 - 1)\sigma_v^2}{\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2} &= \beta_1^{DE} \\
\alpha^2 &= \frac{\beta_1^{DE}}{\sigma_v^2}[\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2] + 1 \\
\alpha^2 - \frac{\beta_1^{DE}}{\sigma_v^2}\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) &= \beta_1^{DE} + 1
\end{aligned}$$

and solving for the α parameter for which the above equality holds. Note that due to the recursive nature of the OC model, $\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) \equiv \breve{\Psi}^i_{t|t-1}(\alpha)$ is itself a function of α.

Appendix C Empirics

Variable	Mnemonic	β_1
Consumer price index	CPI	-0.205***
		(0.067)
Employment	EMP	-0.083
		(0.197)
Housing starts	HOUSING	0.093**
		(0.041)
Industrial production	IP	-0.212***
		(0.045)
Nominal GDP	NGDP	0.113***
		(0.027)
GDP Deflator	PGDP	-0.302***
		(0.045)
Real consumption	RCONSUM	-0.303***
		(0.065)
Real federal government spending	RFEDGOV	-0.240***
		(0.060)
Real GDP	RGDP	-0.169***
		(0.038)
Real nonresidential investment	RNRESIN	-0.180***
		(0.056)
Real residential investment	RRESINV	-0.152**
		(0.074)
Real state/local government spending	RSLGOV	-0.236***
		(0.058)
3-month Treasury bill	TBILL	0.056
		(0.071)
10-year Treasury bond	TBOND	-0.004
		(0.045)
Unemployment rate	UNEMP	0.167**
		(0.068)

Table B1: Pooled Revision Persistence Regressions at $h = 0$, by Variable

Note: The table reports the OLS coefficients from revision persistence regressions across 15 macroeconomic variables reported in the Survey of Professional Forecaters. Column (3) reports the coefficient in front of the revision at the forecaster-level. The revisions are for current period forecasts ($h = 0$). *** denotes 1% significance, ** denotes 5% significance, and * denotes 10% significance.

Appendix D Simulation Results

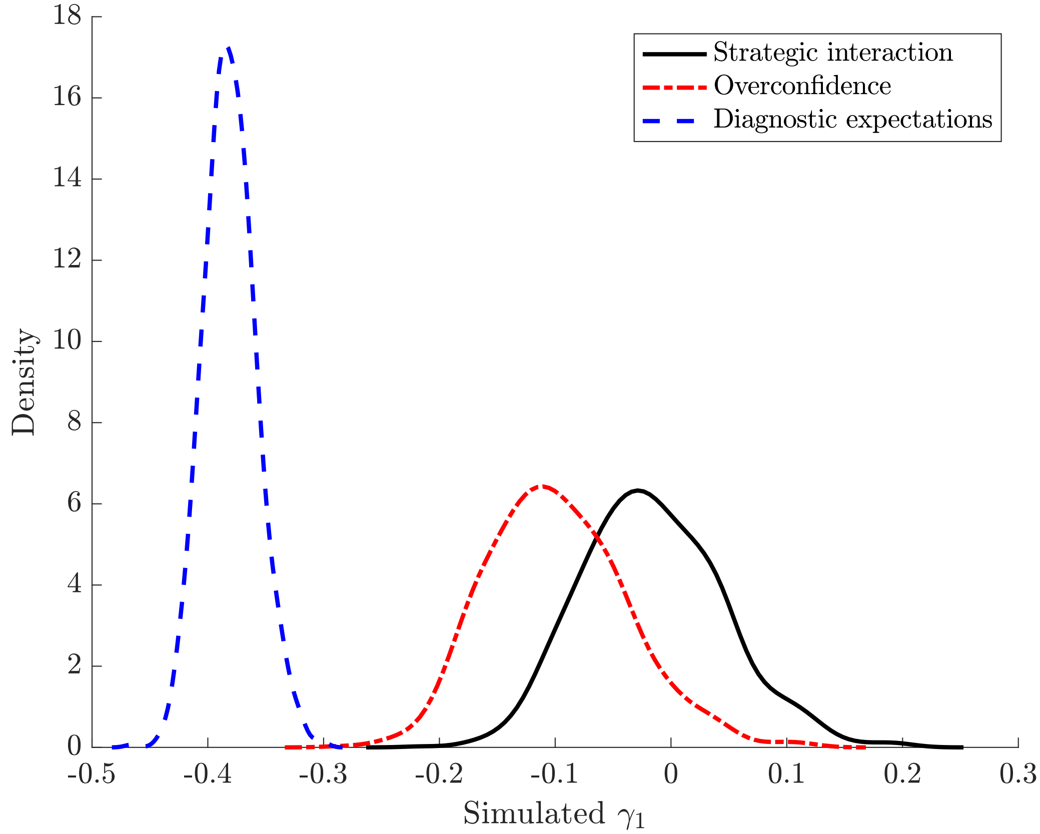

Figure C1: Mapping Over-reactions to Strategic Substitutability

Note: The figure plots the simulated densities of the various γ_1 coefficients across the two models described in the main text and a model of overconfidence.